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On The Determination of Computational Algorithmic Coefficients and Recursive Difference Equations for Control Index Matrices of Single–Delay Linear Neutral Scalar Differential Equations

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Abstract: **This research article determined the structure of the coefficients associated with the optimal computa tional algorithm for control index matrices of single–delay autonomous linear neutral differential equations]. The development of these coefficients exploited the general computational structure of these matrices for positive time periods, skillful assignments of the 0-1 controlling parameters, change of variables techniques, the theory of linear difference equations, and the deployment of deft reasoning to generate easily solvable recursive equations for the coefficients.**

Keywords: **Algorithmic coefficients, Control, Differential Equations, Index, Method, Recursive, Steps.**

1. INTRODUCTION

Control index matrices are integral components of variation of constants formulas in the solutions of terminal function problems in linear and perturbed linear functional differential equations. But quite curiously, no other author has made any serious attempt to investigate the existence or otherwise of their general expressions or to obtain an optimal computational algorithms for various classes of these equations. Effort has usually focused on the single – delay model and the approach has been to start from the interval $[t_1 - h, t_1]$, compute the control index matrices and solutions for given problem instances and then use the method of steps to extend these to the intervals $[t_1 - (j+1)h,\ t_1 - jh]$, for nonnegative integral j, not exceeding 2, for the most part; for real $t_1 : t_1 - (j+1)h > 0.$ Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary j. In other words such approach fails to address the issue of the structure of control index matrices and solutions of terminal function problems quite vital for real-world applications. With a view to addressing such short-comings, [1] blazed the trail by considering the class of double – delay scalar differential equations:

$$\dot{x}(t) = ax(t) + bx(t-h) + cx(t-2h),\ t \in \mathbf{R}, \qquad (1)$$

where a, b and c are arbitrary real constants.

2. METHODS

By deploying ingenious combinations of summation notations, multinomial distribution, greatest integer functions, change of variables techniques, multiple integrals, as well as the method of steps, the paper derived the following optimal expressions for the scalar control index matrices:

$$X(\tau,t_1)=\begin{cases} e^{a(t_1-\tau)},\ \tau\in K_0; & (2)\\ e^{a(t_1-\tau)}+\sum_{i=1}^{j} b^i \frac{\left(t_1-[\tau+ih]\right)^i}{i!} e^{a\left(t_1-[\tau+ih]\right)} & \\ +\sum_{k=1}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{i=0}^{j-2k} b^i c^k \frac{\left(t_1-[\tau+(i+2k)h]\right)^{i+k}}{i!k!} e^{a\left(t_1-[\tau+(i+2k)h]\right)},\ \tau\in K_j,\ j\ge 1 & (3)\end{cases}$$

where $K_j=[t_1-(j+1)h, t_1-jh],\ j\in\{0,1,\cdots\}: t_1-(j+1)h>0; [[.]]$ denotes the greatest integer function, and $X(\tau,t_1)$ denotes a generic control index matrix of the above class of equations for $t\in\mathbf{R}$.

See also [1] for general information on indices of control systems.

[2] obtained a computational algorithm for control index matrices of single-delay autonomous linear neutral differential equations based on transitions of these matrices on contiguous intervals, each of length equal to the delay h.

This article makes further positive contribution to knowledge by using the structure of above algorithm to determine the nature and flavor of the coefficients associated with the control index matrices, thereby considerably reducing the computational effort in [2], as well as eliminating aggregation errors from the resulting components of the control index matrices.

3. RESULTS AND DISCUSSIONS

A careful reflection on (2) and (3) reveals that for $\tau\in K_j,\ j\ge 0, t_1\ne ph,$ for any positive integer p,

$$X(\tau,t_1)=\sum_{k=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{i=0}^{j-2k} b^i c^k \frac{\left(t_1-[\tau+(i+2k)h]\right)^{i+k}}{i!k!} e^{a\left(t_1-[\tau+(i+2k)h]\right)}\operatorname{sgn}(\max\{0, j+1\})$$

$$=\sum_{k=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{i=0}^{j-2k}\ \sum_{(v_0,\cdots,v_{i+k})\in P_{1(i),2(k)}} a_{v_0}\cdots a_{v_{i+k}} \frac{\left(t_1-[\tau+(i+2k)h]\right)^{i+k}}{(i+k)!} e^{a\left(t_1-[\tau+(i+2k)h]\right)}\operatorname{sgn}(\max\{0, j+1\}),\quad (4)$$

where $P_{1(i),2(k)}$ denotes the permutations of the objects 1 and 2 in which 1 occurs i times, 2 k times; $a_0=1,\ a_1=b$ and $a_2=c.$

Let $K_j=\left[t_1-(j+1)h, t_1-jh\right],$ for all nonnegative integral $j: t_1-(j+1)h>0,$ for fixed $t_1>0.$

For $\tau\in K_j,$ let $X(\tau,t_1)=X_j(\tau,t_1)$ be a control index matrix associated with the class of differential equations

$$\dot{x}(t)=a_{-1}\dot{x}(t-h)+a_0x(t)+a_1x(t-h),\qquad (5)$$

on the interval K_j, where

$$X(\tau,t_1)=\begin{cases}1,\ \tau=t_1\\ 0,\ \tau>t_1\end{cases}\qquad (6)$$

Note that $X(\tau,t_1)$ is a generic control index matrix for any $\tau\in\mathbf{R}.$

The coefficients $a_{-1}, a_0,\ a_1$ and the associated functions are all from the real domain. The stage is set for the statement and proof of the first theorem, preliminary to the statement and proof of the major result of this paper.

By careful perusal and exploitation of the results obtained for $X(\tau,t_1)$ on the interval $[t_1-6h,\ t_1-5h]$, [1] successfully devised the following optimal computational algorithm for the control index matrices, with detailed interpretation, without recourse to the class of of differential equations (5) and initial matrix specification (6), using $X_1(\tau,t_1)$ as a starting point.

3.1 A computational Algorithm for transiting from $X_j(\tau,t_1)$ to $X_{j+1}(\tau,t_1),\ j\geq 1$

Let $\tau\in K_j$, let $\lambda_1,\lambda_2\in\{0,1\}$. Suppose that $a_{-1}\left(a_{-1}a_0+a_1\right)\neq 0$. Then

$$X(\tau,t_1)=X_{j+1}(\tau,t_1)=X_1(\tau,t_1)+\sum_{\lambda_1+\lambda_2=1}\sum_{k=1}^{j}a_{-1}^{\lambda_1}\frac{\left(a_{-1}a_0+a_1\right)^{k+\lambda_2}}{(k+\lambda_2)!}\left(t_1-\left(\tau+[k+1]h\right)\right)^{k+\lambda_2}e^{a_0\left(t_1-\left(\tau+[k+1]h\right)\right)} \qquad (7a)$$

$$+\sum_{\lambda_1+\lambda_2=1}\sum_{i=1}^{j-1}a_{-1}^{i+\lambda_1}\frac{\left(a_{-1}a_0+a_1\right)^{1+\lambda_2}}{1+\lambda_2}\left(t_1-\left(\tau+[i+2]h\right)\right)^{1+\lambda_2}e^{a_0\left(t_1-\left(\tau+[i+2]h\right)\right)}\operatorname{sgn}\left(\max\{0,j-1\}\right) \qquad (7b)$$

$$+\sum_{\lambda_1+\lambda_2=1}\sum_{i=1}^{j-2}\sum_{k=2}^{j-i}c_{ik}a_{-1}^{i+\lambda_1}\frac{\left(a_{-1}a_0+a_1\right)^{k+\lambda_2}}{1+\lambda_2+(k-1)\operatorname{sgn}(\lambda_2)}\left(t_1-\left(\tau+[i+k+1]h\right)\right)^{k+\lambda_2}e^{a_0\left(t_1-\left(\tau+[i+k+1]h\right)\right)}\operatorname{sgn}\left(\max\{0,j-2\}\right) \quad (7c)$$

for some real positive constants c_{ik} secured from $X_j(\tau,t_1)$, with the process initiated at $j=1$.

3.2 Remarks on the optimal computational algorithm:

Observe that for $\tau\in K_j, X_{j+1}(\tau,\ t_1)$ can be expressed in the equivalent form

$$X_{j+1}(\tau,\ t_1)=X_1(\tau,\ t_1)$$

$$+\sum_{\lambda_1+\lambda_2=1}\sum_{i=0}^{j-1}\sum_{k=1}^{k-i}c_{ik}a_{-1}^{i+\lambda_1}\frac{\left(a_{-1}a_0+a_1\right)^{k+\lambda_2}}{1+k\operatorname{sgn}(\lambda_2)}\left(t_1-\left(\tau+[i+k+1]h\right)\right)^{k+\lambda_2}e^{a_0\left(t_1-\left(\tau+[i+k+1]h\right)\right)}\operatorname{sgn}\left(\max\{0,j-1\}\right)$$

for some real positive constants c_{ik} secured from $X_j(\tau,\ t_1)$, with the process initiated at $j=1$.

Moreover $c_{0k}=\dfrac{1}{k!}, k\in\{1,2,\cdots,j+1\}$; $c_{i1}=1, i\in\{1,2,\cdots,j\}$ and the transformation from $X_j(\tau,\ t_1)$ to $X_{j+1}(\tau,\ t_1)$ requires only the computations of c_{ik} for $i\in\{1,2,\cdots,j+1-2\}, k\in\{2,3,\cdots,j+1-i\},\ j\geq 3$, such that $i+k=j+1$. Therefore one need only determine $=j-1$ new c_{ik} values, namely

$c_{1\,j}, c_{2\,j-1}, c_{3\,j-2},\cdots,c_{j-1\,2}$.

3.3 Main Result: Theorem on the determination of the structure of the control index matrix coefficients from Algorithm 3.1

Let $j\in\{1,2,\cdots\}$, set $c_{i1}=1$ and $c_{0k}=\dfrac{1}{k!},\ k\in\{1,2,3,\cdots,j+1\}$. Then for $\tau\in K_j, j\in\{2,3,\cdots\}$, $i\in\{1,2,\cdots,j-2\}, k\in\{2,3,\cdots,j-i\}$, the coefficients c_{ik} in algorithm 3.1 are given by:

$$\text{(a)}\ c_{1k}=\frac{1}{(k-1)!}\quad \text{(b)}\ c_{i2}=\frac{1}{2}(i+1)\quad \text{(c)}\ c_{ik}=\frac{1}{k}c_{i\,k-1}+c_{i-1\,k} \qquad (8)$$

Proof

The assignments $\lambda_1=0, \lambda_2=1$, in expression (7a) yield the expression component

$$e^{a_0(t_1-\tau)}+\sum_{k=1}^{j+1}\frac{1}{k!}\left(a_{-1}a_0+a_1\right)^k\left(t_1-[\tau+kh]\right)^k e^{a_0(t_1-[\tau+kh])}$$

$$=e^{a_0(t_1-\tau)}+\sum_{k=1}^{j+1}c_{0\,k}\left(a_{-1}a_0+a_1\right)\left(t_1-[\tau+kh]\right)^k e^{a_0(t_1-[\tau+kh])}\Rightarrow c_{0\,k}=\frac{1}{k!};$$

the assignments $\lambda_1=1,\lambda_2=0,\ j=1,$ in the second component of expression (7a) and $\lambda_1=1,\lambda_2=0,$ in expression (7b) yield the summed component

$$\sum_{i=1}^{[j+1]-1}a_{-1}^i\left(a_{-1}a_0+a_1\right)(t_1-(\tau+[i+1]h)e^{a_0(t_1-(\tau+[i+1]h)}\Rightarrow c_{i1}=1.$$ Therefore $c_{1\,j}$ and $c_{i\,1}$ are well-defined.

$c_{1\,2}$ is obtainable by setting $j=2,\lambda_1=1,\ \lambda_2=0,$ in (7a); $i=1,\lambda_1=0,\lambda_2=1$ in (7b). Therefore $c_{1\,2}=\frac{1}{2!}+\frac{1}{2}=1.$ For $k\geq 3,\ c_{1\,k}$ corresponds to setting $\lambda_1=1,\lambda_2=0$ in (7b); $i=1,\lambda_1=0,\lambda_2=1$ and letting $k\to k-1,$ in summation (7c). Thus $c_{1k}=\frac{1}{k}c_{1\,j-1}+\frac{1}{k!}.$

Proof of (a) in theorem 3.3:

The proof is by mathematical induction on $k.$ Plugging in $i=1,k=2$ on the left side of (6) $\Rightarrow\frac{1}{2}c_{11}+\frac{1}{2!}=\frac{1}{2}+\frac{1}{2!}=1=\frac{1}{(2-1)!}=c_{12}\Rightarrow$ the assertion is valid for $k=2.$ Assume that the assertion is valid for $k\in\{3,\cdots,p\}$, for some integer $p\leq j-2.$ Then from (6) and the induction hypothesis,

$$c_{1\,p+1}=\frac{1}{p+1}c_{1p}+\frac{1}{(p+1)!}=\frac{1}{(p+1)(p-1)!}+\frac{1}{(p+1)!}=\frac{1}{(p+1)(p-1)!}\left(1+\frac{1}{p}\right)$$

$$=\frac{1}{(p+1)(p-1)!}\left(\frac{p+1}{p}\right)=\frac{1}{([p+1]-1)!}.$$

Therefore the assertion (a) is valid for $k=p+1$ and hence valid for all $k\in\{2,3,\cdots,j-1\}$.

(b) For $i\in\{2,3,\cdots,j-2\}$, c_{i2} corresponds to $\lambda_1=0,\lambda_2=1,$ in summation (7b); $\lambda_1=1,\lambda_2=0,$ $k=2,$ and $i\to i-1,$ in summation (7c) $\Rightarrow c_{2\,2}=\frac{1}{2}+c_{1\,2}\Rightarrow c_{1\,2}=\frac{1}{2}+1=\frac{3}{2}=\frac{1}{2}(2+1)\Rightarrow$ assertion (b) of theorem 3.3 is valid for $i=2.$ Assume that the assertion is valid $i\in\{3,\cdots,p\}$ for some $p\in\{3,\cdots,j-1\}$. Then by (6) and the induction hypothesis, $c_{i2}=c_{i-1\,2}+\frac{1}{2}$ $=\frac{1}{2}([i-1+1])+\frac{1}{2}=\frac{1}{2}(i+1)\Rightarrow$ assertion (b) of theorem 3.3 is valid for $p=j;$ so (b) of theorem 3.3 is valid.

(c) For $i\in\{2,3,\cdots,j-2\}$ and $k\in\{3,\cdots,j-1\}$, summations (7a) and (7b) are not feasible. Therefore only (7c) is applicable in the determination of $c_{i\,j}$, in which case $c_{i\,k}$ corresponds to letting $\lambda_1=1,\lambda_2=0,\ i\to i-1$ in summation (7c), returning to the parent (7c), then letting $\lambda_1=0,\lambda_2=1$ and $k\to k-1,$ in summation (7c) and adding up the results from both contingencies to obtain $c_{ik}=\frac{1}{k}c_{i\,k-1}+c_{i-1\,k}$; $i\in\{2,3,\cdots,j-2\}$, $k\in\{3,\cdots,j-i\}$. Moreover

$c_{i\,2}-\frac{1}{2}c_{i\,1}=\frac{1}{2}(i+1)-\frac{1}{2}=\frac{i}{2}=\frac{([i-1]+1)}{2}=c_{i-1\,2}\Rightarrow$ the assertion is also valid for $k=2$, completing the proof.

3.4 Corollary on the coefficients:

(a) There are exactly $\sum_{i=2}^{j-2}\sum_{k=2}^{j-i}1=(j-3)+(j-4)+\cdots+1=\frac{1}{2}(j-3)(j-2)$ constants to be determined from the recursive equations.

(b) In the transition from $X_j(\tau,t_1)$ to $X_{j+1}(\tau,t_1)$, there are exactly

$\frac{1}{2}\left[([j+1]-3)([j+1]-2)-(j-3)(j-2)\right]=j-2$ new coefficients to be determined;

the other $\frac{1}{2}(j-3)(j-2)$ are picked up from $X_j(\tau,t_1)$.

The implication of (b) in corollary 3.4 is far-reaching: the number of coefficients to be obtained and quite easily too, is pruned to a mere size of $k-2$, precisely those $c_{i\,j}$s for which $i+j=k+1$, namely $c_{2\,k-1},c_{3\,k-2},\cdots,c_{k-1\,2}$; needless to say that $c_{0\,k+1}=\frac{1}{(k+1)!}$, $c_{k1}=1$ and $c_{1k}=\frac{1}{(k-1)!}$.

3.5 Illustrative Examples:

For $\tau\in\bigcup_{j=0}^{7}K_j$, the expressions for $X(\tau,t_1)$ were determined in [1]. Now the consistency of theorem 3.3 with algorithm 3.1 will be demonstrated for $\tau\in\bigcup_{j=6}^{7}K_j$. Furthermore, theorem 3.3 will be used to extend the expressions for $X(\tau,t_1)$ to the τ-interval K_8.

To secure $X(\tau,t_1)$, for $\tau\in K_6$, only the additional coefficients c_{ik}: $i+k=6$ need to be determined, namely $c_{06},c_{15},c_{51},c_{15},c_{51},c_{24},c_{42}$ and c_{33}. The remaining coefficients can be picked up from $X(\tau,t_1)$, for $\tau\in K_5$.

Now $c_{06}=\frac{1}{6!}$, $c_{15}=\frac{1}{5!}$, $c_{51}=1$; $c_{24}=\frac{1}{4}c_{23}+c_{14}=\frac{1}{4}(1)+\frac{1}{3!}=\frac{5}{12}$, $c_{42}=\frac{1}{2}c_{41}+c_{32}$ $=\frac{1}{2}+2=\frac{5}{2}$, $c_{33}=\frac{1}{3}c_{32}+c_{23}=\frac{1}{3}(2)+1=\frac{5}{3}$. Hence

$$X(\tau,t_1)=e^{a_0(t_1-\tau)}+\left(\sum_{k=1}^{6}\frac{(a_{-1}a_0+a_1)^k}{k!}(t_1-[\tau+kh])^i e^{a_0(t_1-[\tau+kh])}\right)$$

$$+\sum_{i=1}^{6-1}a_{-1}^i(a_{-1}a_0+a_1)(t_1-(\tau+[i+1]h))e^{a_0(t_1-(\tau+[i+1]h))}+a_{-1}(a_{-1}a_0+a_1)^2(t_1-[\tau+3h])^2e^{a_0(t_1-[\tau+3h])}$$

$$+\left[a_{-1}(a_{-1}a_0+a_1)^3\frac{(t_1-[\tau+4h])^3}{2}+\frac{3}{2}a_{-1}^2(a_{-1}a_0+a_1)^2(t_1-[\tau+4h])^2\right]e^{a_0(t_1-[\tau+4h])}$$

$$+\left[\begin{array}{l}a_{-1}(a_{-1}a_0+a_1)^4\frac{(t_1-[\tau+5h])^4}{3!}+a_{-1}^2(a_{-1}a_0+a_1)^3(t_1-[\tau+5h])^3\\+2a_{-1}^3(a_{-1}a_0+a_1)^2(t_1-[\tau+5h])^2\end{array}\right]e^{a_0(t_1-[\tau+5h])}$$

$$+\begin{bmatrix} a_{-1}\left(a_{-1}a_0+a_1\right)^5\dfrac{\left(t_1-[\tau+6h]\right)^5}{4!}+\dfrac{5}{12}a_{-1}^2\left(a_{-1}a_0+a_1\right)^4\left(t_1-[\tau+6h]\right)^4 \\ +\dfrac{5}{2}a_{-1}^4\left(a_{-1}a_0+a_1\right)^2\left(t_1-[\tau+6h]\right)^2+\dfrac{5}{3}a_{-1}^3\left(a_{-1}a_0+a_1\right)^3\left(t_1-[\tau+6h]\right)^3 \end{bmatrix} e^{a_0\left(t_1-[\tau+6h]\right)}, \tau\in K_6.$$

For $\tau\in K_7$, the additional coefficients are $c_{07}, c_{16}, c_{61}, c_{25}, c_{52}, c_{34}\, c_{43}$:

$$c_{07}=\frac{1}{6!}, c_{16}=\frac{1}{5!},\ c_{61}=1,\ c_{25}=\frac{1}{5}c_{24}+c_{15}=\frac{1}{5}\left(\frac{5}{12}\right)+\frac{1}{4!}=\frac{1}{8}, c_{52}=\frac{1}{2}c_{51}+c_{42}=\frac{1}{2}(1)+\frac{5}{2}=3,$$

$$c_{34}=\frac{1}{4}c_{33}+c_{24}=\frac{1}{4}\left(\frac{5}{3}\right)+\frac{5}{12}=\frac{5}{6}\text{ and } c_{43}=\frac{1}{3}c_{42}+c_{33}=\frac{1}{3}\left(\frac{5}{2}\right)+\frac{5}{3}=\frac{5}{2}.$$ Hence

$$X(\tau,t_1)=e^{a_0(t_1-\tau)}+\left(\sum_{k=1}^{7}\frac{\left(a_{-1}a_0+a_1\right)^k}{k!}\left(t_1-[\tau+kh]\right)^i e^{a_0\left(t_1-[\tau+kh]\right)}\right)$$

$$+\sum_{i=1}^{7-1}a_{-1}^i\left(a_{-1}a_0+a_1\right)(t_1-(\tau+[i+1]h))e^{a_0(t_1-(\tau+[i+1]h))}+a_{-1}\left(a_{-1}a_0+a_1\right)^2(t_1-[\tau+3h])^2e^{a_0(t_1-[\tau+3h])}$$

$$+\left[a_{-1}\left(a_{-1}a_0+a_1\right)^3\frac{(t_1-[\tau+4h])^3}{2}+\frac{3}{2}a_{-1}^2\left(a_{-1}a_0+a_1\right)^2(t_1-[\tau+4h])^2\right]e^{a_0(t_1-[\tau+4h])}$$

$$+\begin{bmatrix} a_{-1}\left(a_{-1}a_0+a_1\right)^4\dfrac{\left(t_1-[\tau+5h]\right)^4}{3!}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^3\left(t_1-[\tau+5h]\right)^3 \\ +2a_{-1}^3\left(a_{-1}a_0+a_1\right)^2\left(t_1-[\tau+5h]\right)^2 \end{bmatrix} e^{a_0\left(t_1-[\tau+5h]\right)}$$

$$+\begin{bmatrix} a_{-1}\left(a_{-1}a_0+a_1\right)^5\dfrac{\left(t_1-[\tau+5h]\right)^5}{4!}+\dfrac{5}{12}a_{-1}^2\left(a_{-1}a_0+a_1\right)^4\left(t_1-[\tau+6h]\right)^4 \\ +\dfrac{5}{2}a_{-1}^4\left(a_{-1}a_0+a_1\right)^2\left(t_1-[\tau+6h]\right)^2+\dfrac{5}{3}a_{-1}^3\left(a_{-1}a_0+a_1\right)^3\left(t_1-[\tau+6h]\right)^3 \end{bmatrix} e^{a_0\left(t_1-[\tau+6h]\right)}$$

$$+\begin{bmatrix} a_{-1}\left(a_{-1}a_0+a_1\right)^6\dfrac{\left(t_1-[\tau+7h]\right)^6}{5!}+\dfrac{1}{8}a_{-1}^2\left(a_{-1}a_0+a_1\right)^5\left(t_1-[\tau+7h]\right)^5 \\ +3a_{-1}^5\left(a_{-1}a_0+a_1\right)^2\left(t_1-[\tau+7h]\right)^2+\dfrac{5}{6}a_{-1}^3\left(a_{-1}a_0+a_1\right)^4\left(t_1-[\tau+7h]\right)^4 \\ +\dfrac{5}{2}a_{-1}^4\left(a_{-1}a_0+a_1\right)^3\left(t_1-[\tau+7h]\right)^3 \end{bmatrix} e^{a_0\left(t_1-[\tau+7h]\right)}, \tau\in K_7.$$

The consistency of theorem 3.3 with algorithm 3.1 is verified for $\tau\in K_7$.

Finally, for $\tau\in K_8$, the additional coefficients are $c_{08}, c_{17}, c_{71}, c_{26}, c_{62}, c_{35}, c_{53}, c_{44}$:

$$c_{08}=\frac{1}{7!}, c_{17}=\frac{1}{6!},\ c_{71}=1,\ c_{26}=\frac{1}{6}c_{25}+c_{16}=\frac{1}{6}\left(\frac{1}{8}\right)+\frac{1}{5!}=\frac{7}{240}, c_{62}=\frac{1}{2}c_{61}+c_{52}=\frac{1}{2}(1)+3=\frac{7}{2},$$

$$c_{35}=\frac{1}{5}c_{34}+c_{25}=\frac{1}{5}\left(\frac{5}{6}\right)+\frac{1}{8}=\frac{7}{24},\ c_{53}=\frac{1}{3}c_{52}+c_{43}=\frac{1}{3}(3)+\frac{5}{2}=\frac{7}{2}, c_{44}=\frac{1}{4}c_{43}+c_{34}=\frac{1}{4}\left(\frac{5}{2}\right)+\frac{5}{6}=\frac{35}{24}.$$

Hence

$$X(\tau,t_1)=e^{a_0(t_1-\tau)}+\left(\sum_{k=1}^{8}\frac{(a_{-1}a_0+a_1)^k}{k!}\left(t_1-[\tau+kh]\right)^i e^{a_0(t_1-[\tau+kh])}\right)$$

$$+\sum_{i=1}^{8-1}a_{-1}^i(a_{-1}a_0+a_1)(t_1-(\tau+[i+1]h))e^{a_0(t_1-(\tau+[i+1]h))}+a_{-1}(a_{-1}a_0+a_1)^2(t_1-[\tau+3h])^2e^{a_0(t_1-[\tau+3h])}$$

$$+\left[a_{-1}(a_{-1}a_0+a_1)^3\frac{(t_1-[\tau+4h])^3}{2}+\frac{3}{2}a_{-1}^2(a_{-1}a_0+a_1)^2(t_1-[\tau+4h])^2\right]e^{a_0(t_1-[\tau+4h])}$$

$$+\left[\begin{array}{l}a_{-1}(a_{-1}a_0+a_1)^4\frac{(t_1-[\tau+5h])^4}{3!}+a_{-1}^2(a_{-1}a_0+a_1)^3(t_1-[\tau+5h])^3\\+2a_{-1}^3(a_{-1}a_0+a_1)^2(t_1-[\tau+5h])^2\end{array}\right]e^{a_0(t_1-[\tau+5h])}$$

$$+\left[\begin{array}{l}a_{-1}(a_{-1}a_0+a_1)^5\frac{(t_1-[\tau+6h])^5}{4!}+\frac{5}{12}a_{-1}^2(a_{-1}a_0+a_1)^4(t_1-[\tau+6h])^4\\+\frac{5}{2}a_{-1}^4(a_{-1}a_0+a_1)^2(t_1-[\tau+6h])^2+\frac{5}{3}a_{-1}^3(a_{-1}a_0+a_1)^3(t_1-[\tau+6h])^3\end{array}\right]e^{a_0(t_1-[\tau+6h])}$$

$$+\left[\begin{array}{l}a_{-1}(a_{-1}a_0+a_1)^6\frac{(t_1-[\tau+7h])^6}{5!}+\frac{1}{8}a_{-1}^2(a_{-1}a_0+a_1)^5(t_1-[\tau+7h])^5\\+3a_{-1}^5(a_{-1}a_0+a_1)^2(t_1-[\tau+7h])^2+\frac{5}{6}a_{-1}^3(a_{-1}a_0+a_1)^4(t_1-[\tau+7h])^4\\+\frac{5}{2}a_{-1}^4(a_{-1}a_0+a_1)^3(t_1-[\tau+7h])^3\end{array}\right]e^{a_0(t_1-[\tau+7h])}$$

$$+\left[\begin{array}{l}a_{-1}(a_{-1}a_0+a_1)^7\frac{(t_1-[\tau+8h])^7}{6!}+\frac{7}{240}a_{-1}^2(a_{-1}a_0+a_1)^6(t_1-[\tau+8h])^6\\+\frac{7}{2}a_{-1}^6(a_{-1}a_0+a_1)^2(t_1-[\tau+8h])^2+\frac{7}{24}a_{-1}^3(a_{-1}a_0+a_1)^5(t_1-[\tau+8h])^5\\+\frac{7}{2}a_{-1}^5(a_{-1}a_0+a_1)^3(t_1-[\tau+8h])^3+\frac{7}{2}a_{-1}^5(a_{-1}a_0+a_1)^3(t_1-[\tau+8h])^3\\+\frac{35}{24}a_{-1}^4(a_{-1}a_0+a_1)^4(t_1-[\tau+8h])^4\end{array}\right]e^{a_0(t_1-[\tau+8h])},\ \tau\in K_8.$$

For the case $a_{-1}(a_{-1}a_0+a_1)=0,\ X(\tau,t_1)=e^{a_0(t_1-\tau)}\max\{j+1,0\},\ j\in\{0,1,\cdots\}$

4. CONCLUSION

This article obtained the structure of the coefficients of the control index matrices of single-delay neutral differential equations in [1] through exact determination of some of those and the derivation of easily solvable recursive difference equation for the remaining coefficients, proving conclusively that there is no general expression for such coefficients. This contrasts quite sharply with the coefficients of control index matrices of single-delay and the class of double-delay differential equations whose expressions are clearly established, as in expressions (2) and (3).

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